1. Introduction

In this paper a certain integrodifferential equations are proposed for the solution of EM problems with 3D dielectric bounded scatterers. This article can be viewed as an extension of our previous work [1]. It is also necessary to point out that Waterman’s [2] and Kyurkchan’s [3] methods serve as a background of this work.

2. Differential Formulation

Suppose that certain EM field with strength $E_0, H_0$ and time dependence $e^{-i\omega t}$ propagates in linear isotropic uniform medium. Let the scatterer bounded with the smooth surface $S$ occupies the finite volume of space $\Omega_2 = G_2 \cup S$. Complement of $\Omega_2$ to entire space $\Omega^1$ we denote as $G_1 = \Omega^1 / \Omega_2$ ( $G_i = G_i \cup S$ ). Interaction of the incident field with an inhomogeneity results in the appearance of scattered and interior fields which we denote as $E_1, H_1$ and $E_2, H_2$, respectively. We shall refer to the properties of the surrounding medium by utilizing script “1” and script “2” will be used for denoting the constants of the material filling the body. By means of Maxwell’s equations and classical constitutive relations one can show that the fields $E_j, H_j$ ($j=1,2$) satisfy the uniform Helmholtz equations. On the surface $S$ the tangential parts of field vectors must be equated:

$$\frac{\partial}{\partial n} \left( \frac{\partial}{\partial n} \vec{E}_0 + \vec{E}_1 \right) = \left[ \vec{n} \times \vec{H}_0 \right]_S = \left[ \vec{n} \times \vec{H}_1 \right]_S. \quad (1)$$

The denotation $\vec{n}$ corresponds to the unit normal to $S$ pointing outward the volume of the scatterer $G_2$. We shall also use the denotation $\vec{n}$ that corresponds to the normal pointing inside the body. Additionally, the vectors $\vec{E}_1, \vec{H}_1$ must satisfy the Silver – Muller radiation conditions [2].

3. Stratton-Chu Integrals for Interior and Exterior Field

The starting point to yield Stratton-Chu representations is the integration of Helmholtz equations over the volume $\Omega_j$ in the following way:

$$\int (\vec{\nabla} \cdot \vec{\nabla} \tilde{F}_j) \cdot (\vec{a} \vec{g}_j) \, dV + \left[ \int k^2 \tilde{F}_j \cdot (\vec{a} \vec{g}_j) \, dV = 0, \right.$$

where $\tilde{F}_j = \vec{E}_j, \vec{H}_j; j=1,2$; $\vec{a}$ is the arbitrary constant vector, $\vec{g}_j = \exp(ik_j \vec{r} - \vec{r}')/4\pi |\vec{r} - \vec{r}'|$ denotes free space Green’s function. By utilizing several identities of vector analysis and Gauss theorem one may define the following integral representations:

$$\left[ \int_{\Omega_j} \delta(|\vec{r} - \vec{r}'|) \, dV = -\int_{\Omega_j} \left[ (\vec{n} \cdot \vec{E}_2) \vec{V} + (\vec{n} \times \vec{E}_2) \times \vec{V} + ik_2 Z_2 (\vec{n} \times \vec{H}_2) \right]_S g_2 \, dS, \right.$$ \hspace{1cm} \left(2\right)$$

$$\left[ \int_{\Omega_j} \delta(|\vec{r} - \vec{r}'|) \, dV = -\int_{\Omega_j} \left[ (\vec{n} \cdot \vec{H}_2) \vec{V} + (\vec{n} \times \vec{H}_2) \times \vec{V} - i(k_2 Z_2) (\vec{n} \times \vec{E}_2) \right]_S g_2 \, dS, \right.$$ \hspace{1cm} \left(2\right)$$

where $\delta(|\vec{r} - \vec{r}'|)$ denotes Dirac delta. Evidently, the exact form of the l.h.s. in (2) depends on the concrete location of the observation point $\vec{r}'$ in space. Two situations are possible: if $\vec{r}' \in \Omega_2$ then
the integral on the l.h.s. presents electric field at location \( \vec{r}' \), otherwise (\( G_r' \in \vec{r}' \)) it will be equal to zero. In relation (2) we may carry differential operators of observer to source points as well as take curl operator of both sides of the equality. By utilizing the identities 
\[
\hat{\nabla}' \times \nabla' \times (\hat{\nabla}_2 g_2) = r'^{-1} \hat{\nabla}' \times (\hat{\nabla}_1 g_1) \quad \text{and} \quad \hat{\nabla}' \times \nabla' \times (\hat{\nabla}_2 g_2) = r'^{-1} \hat{\nabla}' \times (\hat{\nabla}_1 g_1),
\]
[4] one finds the expression for the radial components of EM vectors (magnetic component is only shown for the sake of brevity):
\[
\begin{align*}
\int \int \int \left( \hat{\nabla}' \times \hat{\nabla} - \hat{\nabla}' \times \hat{\nabla}' \right) dV = \frac{1}{k_2 Z_j} \int \left( \hat{\nabla}_1 g_1 \right) dS - \frac{1}{r} \int \left( \hat{\nabla}_2 g_2 \right) dS
\end{align*}
\]
where \( \tilde{K}_1 \equiv \hat{\nabla}' \times (\hat{\nabla}_1) \) and \( \tilde{K}_2 \equiv \hat{\nabla}' \times (\hat{\nabla}_2) \). Analogous expressions of field vectors we may get for exterior space:
\[
\begin{align*}
\int \int \int \left( \hat{\nabla}' \times \hat{\nabla} - \hat{\nabla}' \times \hat{\nabla}' \right) dV = \frac{1}{k_2 Z_j} \int \left( \hat{\nabla}_2 g_2 \right) dS - \frac{1}{r} \int \left( \hat{\nabla}_1 g_1 \right) dS.
\end{align*}
\]

4. Integrodifferential System of Equations and Algebraic Problem

If we desire to yield constitutive integrodifferential equations by means of (3) and (4) that allow obtaining unknown EM fields we need suitable operator forms of vectors \( \hat{E}_j, \hat{H}_j \). For free space we choose the following differential relations:
\[
\hat{E}_j[u,v] = \tilde{K}_2[v] + i k_2 Z_j \tilde{K}_1[u], \quad \hat{H}_j[u,v] = \tilde{K}_2[u] - i(k_2 Z_j) \tilde{K}_1[v],
\]
where scalar fields \( u \) and \( v \) are expressed by the Atkinson-Wilcox series [2]. These representations satisfy “uniform” Maxwell’s equations (free of volume currents on the r.h.s.) as well as Silver-Muller radiation conditions.

Before we choose the operator form of interior field let us speak out the following consideration. Physics of the solution requires that analytical representation being used has to be free of singularities inside the scatterer. In simplest case, we can get this result by using vector spherical harmonics expansion which radial multipliers contain the Bessel function of the first kind \( j_n \). Therefore, in analogy with (5) one yields:
\[
\begin{align*}
\hat{E}_2[a,b] &= \tilde{K}_2[b] + i k_2 Z_j \tilde{K}_1[a],
\hat{H}_2[a,b] &= \tilde{K}_2[a] - i(k_2 Z_j) \tilde{K}_1[b],
\end{align*}
\]
thereby \( a \) and \( b \) functions are expressed by means of the series:
\[
\begin{align*}
\left\{ \begin{array}{c}
a(\vec{r}) \\
b(\vec{r})
\end{array} \right\} &= \sum_{nm} \left\{ \begin{array}{c}
a_{nm} \\
b_{nm}
\end{array} \right\} \left( -i \right)^{n+1} j_n(k_2 r) Y_{nm}(\theta, \phi),
\end{align*}
\]
where \( Y_{nm}(\theta, \phi) = P_{nm}^m(\cos \theta) e^{im \phi} \) — spherical functions. Consider equalities (3) and (4). As a first step we perform the “exchange” by boundary quantities of EM field in the integrand terms according to boundary conditions (2) [3,5]. After that we put the observation point \( \vec{r}' \) in far zone. Transform first equation in (4). It is evident that in this case we have to choose the nonzero magnitude of the integral on the l.h.s. since the point \( \vec{r}' \) is placed in \( G_i \). Furthermore, in far zone we yield asymptotic forms of Green’s function and Atkinson-Wilcox series for the radial component of the magnetic vector \( \hat{H}_{1r} \) as follows [1]:
\[
\begin{align*}
g_1 &= e^{ik_1 r'} f_j + O\left( r'^{-2} \right), \\
H_{1r} &= -\frac{e^{ik_1 r'} B_{10}(\theta', \phi')}{r} + O\left( r'^{-3} \right),
\end{align*}
\]
where \( f_j = e^{-ik_1 r} \) and \( B \) denotes Beltrami operator [2]. By substituting latter expressions in (4) we come to the integrodifferential equation:
\[
\begin{align*}
\frac{1}{4\pi Z_1} \int \left( \hat{\nabla}_1 g_1 \right) dS - \frac{k_1}{4\pi} \int \left( \hat{\nabla}_2 g_2 \right) dS
\end{align*}
\]
Now we consider integral (3) more explicitly. Since observation point we hold in \( G_i \) we need to take the zero magnitude of the integral on the l.h.s. in (3) [5]. By utilizing the asymptotic form of Green’s function and rendering identical transforms one yields the equation [5]:
\[
\begin{align*}
0 &= \frac{1}{4\pi Z_2} \int \left( \hat{\nabla}_1 g_1 \right) dS - \frac{k_2}{4\pi} \int \left( \hat{\nabla}_2 g_2 \right) dS.
\end{align*}
\]
where \( f_2 = e^{ikz^2 \hat{z}} \). Another one pair of integro-differential equations we may define by the use of Stratton-Chu integrals expressing electric vectors. Taking into account the construction of EM field operators in the integrand terms one can deduce that equations (7)-(8) present integrodifferential system with respect to the unknown functions \( u_0(\theta', \varphi') \), \( v_0(\theta', \varphi') \) [1] and algebraic coefficients \( a_{nm} \) and \( b_{nm} \) as well. Analysis of the structure of equations (7) - (8) leads us to the following inference: equation (7) presents generalization of the equation defined in [1] and another equation is intrinsic to the null-field method [2].

Finally integrodifferential system (7)-(8) can be reduced to pure algebraic one by expanding \( u_0 \) and \( v_0 \) functions into Fourier series with respect to spherical harmonics basis as it has been done in [1]. Thus we get the algebraic analog of (7) – (8) as:

\[
\vec{w}_l = \hat{M}_l \vec{w}_l, \quad \hat{M}_l \vec{w}_l = -\vec{V}.
\]

It should be mentioned that the first equation in (9) coincides with the same in [3]. A few words have to be said regarding the solution algorithm that results from the construction of algebraic system (9). The second equation in (9) does not contain the vector of unknowns \( \vec{w}_2 \), hence, these two linear systems can be solved consequently. In fact, in many cases the definition of interior field is not obligatory. For instance, in problems with dielectric resonator antennas designs, the quantities characterizing their efficiency (radiation patterns, spectral responses) can be found via exterior field. Thus, the computational complexity of the algorithm differs slightly compared to a perfectly conducting body problem [1].

5. Validation

The aim which we pursue in this section is to verify equations (7)-8) by the solution of well-known canonical problems. We begin with the analytical solution for dielectric sphere excited by plane wave. Thus, let linearly polarized plane wave

\[
\vec{E}_0 = E_0 \hat{e}_x e^{ikx \hat{x}}, \quad \vec{H}_0 = (E_0/Z_1) \hat{e}_y e^{iky \hat{y}}
\]

propagates in free space along the \( z \) axis in positive direction. Let also dielectric sphere of radius \( R \) is placed in space so that its center coincides with the origins of Cartesian and spherical coordinate systems. By utilizing known identities of spherical analysis [6] it is easy to show that the equations (9) result in the form:

\[
\left( ik\frac{Z_1}{Z_2} \hat{M}_2 F_{12} + ik_1 \hat{M}_1 F_{21} \right) \vec{u} = \vec{u}, \quad \left( ik\frac{Z_1}{Z_2} \hat{M}_2 F_{12} + ik_1 \hat{M}_1 F_{21} \right) \vec{v} = -\vec{V} \hat{u},
\]

\[
\left( ik\frac{Z_1}{Z_2} \hat{M}_2 F_{12} + ik_1 \hat{M}_1 F_{21} \right) \vec{v} = \vec{v}, \quad \left( ik\frac{Z_1}{Z_2} \hat{M}_2 F_{12} + ik_1 \hat{M}_1 F_{21} \right) \vec{u} = -\vec{V} \hat{v},
\]

The elements of \( \hat{M}_l \) are written as \( M_{pqnmab} = \frac{i}{2\pi n(n+l)} \int_S \left( \vec{n} \times \vec{K}_p [F_{ab}(k_j, \vec{r})] \right) \cdot \vec{K}_q \left( T_{nm}(k_j, \vec{r}) \right) dS \),

\[
M_{pqnmab} = \frac{1}{2\pi n(n+l)} \int_S \left( \vec{n} \times \vec{K}_p [F_{ab}(k_j, \vec{r})] \right) \cdot \vec{K}_q \left( T_{nm}(k_j, \vec{r}) \right) dS, \quad \text{where} \quad T_{nm}(k, \vec{r}) = i^{n+l} h_n^{(1)}(kr) \ Y_{nm}(\theta, \varphi),
\]

\( F_{nm}(k, \vec{r}) = (-i)^{n+l} j_n(kr) Y_{nm}(\theta, \varphi). \) The elements of the vectors \( \vec{V}^u = \{ V_{nm}^u \} \), \( \vec{V}^v = \{ V_{nm}^v \} \) have the form

\[
V_{nm}^u = \frac{1}{2\pi n(n+l)} \left( \frac{Z_1}{Z_2} \right) \int_S \left( \vec{n} \times \vec{E}_0 \right) \cdot \vec{K}_2 \left( F_{nm}(k_2, \vec{r}) \right) dS + ik_2 \int_S \left( \vec{n} \times \vec{H}_0 \right) \cdot \vec{K}_1 \left( F_{nm}(k_2, \vec{r}) \right) dS,
\]

\[
V_{nm}^v = \frac{1}{2\pi n(n+l)} \left( -Z_2 \right) \int_S \left( \vec{n} \times \vec{E}_0 \right) \cdot \vec{K}_2 \left( F_{nm}(k_2, \vec{r}) \right) dS + ik_2 \int_S \left( \vec{n} \times \vec{H}_0 \right) \cdot \vec{K}_1 \left( F_{nm}(k_2, \vec{r}) \right) dS.
\]

By utilizing the system of integral identities from [6] it can be easily proved that the system (11) results in the well-known Mie solution [6].

Let us go in for searching the numerical solutions. In our investigation the diffraction of normally incident plane wave on dielectric cubes of different size and permittivity has been considered. The results have been compared against known literature data whenever possible. The 2d and 3d RCS images will be shown during presentation. Herein we study the convergence of the algorithm and its time expenses for the following example. Consider diffraction of plane wave on a cube having 0.5 meter on a side. In this instance we shall keep frequency unchanged (\( f \approx 300 \text{ MHz} \),

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only refractive index $n$ will be altered within a range $3 \div 8$. The criterion of convergence will be a minimum of relative error: $\delta_N = |P_N - P_{N-1}| / \min(P_{N-1}, P_N)$, where $P_N$ denotes radiated power [1]. Subscript $N$ shows the highest order of spherical harmonics which we hold in the truncated Fourier series. The results are accumulated in Table 1. The columns CPU1 and CPU2 denote the time of algebraic system forming and solution, respectively, for different orders $N$ and magnitudes of refractive index. As seen, in all the cases system forming time greatly exceeds a solution time, though we used most time expensive solver based on Gaussian elimination. Note that algebraic system forming time is mainly determined by the calculation time of r.h.s. and l.h.s. integrals in (9). This time, obviously, depends on two issues: the number of harmonics which we hold in the truncated series and the discretization order of cube surface used in numerical integration. The discretization order has been found separately for each case shown in Table 2 in numerical tests; the cube surface has been subdivided into square patches. The second order quadrature formula (four nodes per patch) has been applied within a patch. The discretization order as a function of refractive index is shown in Table II, where the parameter $N_{pe}$ denotes the number of patch elements along arbitrary edge notwithstanding its spatial orientation. The data shown in the tables result in the inference: no problems have been observed with the objects having high contrast dielectric filling with respect to the environment.

**Table 1: Relative errors and time of computation**

<table>
<thead>
<tr>
<th>Order $N$</th>
<th>$\delta_N$</th>
<th>CPU1, sec</th>
<th>CPU2, sec</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n = 4.0$</td>
<td>$10/16.9/33 ./1.$</td>
<td>$13/7/125./4.$</td>
<td>$16/9.2/376./11.$</td>
</tr>
<tr>
<td>$n = 5.0$</td>
<td>$11/1./47./1.6$</td>
<td>$14/2.95/164./6.$</td>
<td>$17/1.6/474./16.$</td>
</tr>
<tr>
<td>$n = 6.0$</td>
<td>$12/89/65./2.5$</td>
<td>$15/.52/212./8.$</td>
<td>$18/1./585./22.$</td>
</tr>
<tr>
<td>$n = 7.0$</td>
<td>$13/06/88./3.8$</td>
<td>$16/.43/266./11.$</td>
<td>$19/.25/703./29.$</td>
</tr>
</tbody>
</table>

**Table 2: Discretization order versus refractive index**

<table>
<thead>
<tr>
<th>$N_{pe}$</th>
<th>$n$</th>
<th>$N_{pe}$</th>
<th>$n$</th>
<th>$N_{pe}$</th>
<th>$n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>3.0</td>
<td>12</td>
<td>5.0</td>
<td>16</td>
<td>7.0</td>
</tr>
<tr>
<td>10</td>
<td>4.0</td>
<td>14</td>
<td>6.0</td>
<td>18</td>
<td>8.0</td>
</tr>
</tbody>
</table>

**References**


