Variational Expressions of Capacitance of An Annular Ring on A Dielectric Substrate

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1. Introduction

Microstrip annular rings have been used extensively for the resonator, filter, antenna element and RFID. A variational expression for capacitance of annular ring was derived using the charge density on the microstrip ring [1], and it provides a lower bound of the exact value. The upper bound as well as the lower bound is required to estimate the range of error. In this paper, the electric fields on the cylindrical aperture surface are introduced for the source quantities, and the variational expression is derived with respect to this aperture field and it provides an upper bound of the exact value. The accurate calculations of upper and lower bound capacitances are performed by using the extended version of spectral domain approach (ESDA) with taking the edge effects in account [2].

2. Analytical Procedure

2.1 Derivation of the upper bound expression of capacitance

Fig. 1 shows the microstrip annular ring with the substrate of the dielectric constant $\varepsilon_r$ and the thickness $d$. The width and radius of annular ring are $W$ and $R$, respectively. The upper ground conductor is placed at $h$ above the ring. The ring and ground conductors are assumed to be perfect conductors (PEC), and the microstrip ring has zero metallization thickness.
It is assumed that fields have no circumferential components \( E_z = H_z = 0 \) and no variation in \( \phi \). Capacitance of the annular ring is calculated by using ESDA.

In the calculation process, first the aperture fields are introduced at the circular cylindrical aperture surfaces between the ring and the ground conductors at \( \rho = R \), \( e_{l (II)} \) and \( e_{l (III)} \), and at \( \rho = R + W \), \( e_{z (II)} \) and \( e_{z (III)} \), as shown in Fig. 1. Then, the whole region is divided into sub-regions; region (I) \( (\rho \leq R, -d \leq z \leq h) \), region (II) \( (R \leq \rho \leq R + W, 0 \leq z \leq h) \), region (III) \( (R \leq \rho \leq R + W, -d \leq z \leq 0) \) and region (IV) \( (R + W \leq \rho, -d \leq z \leq h) \). Applying the equivalence theorem, each subregion can be treated independently. Electric fields in region (i) are expressed in terms of the potential functions as

\[
E^{(i)} = -\nabla_i \psi^{(i)} (\rho, z)
\]  

where \( \nabla_i \) is the transverse del operator,

\[
\nabla_i = i_\rho \frac{\partial}{\partial \rho} + i_z \frac{\partial}{\partial z}
\]

The potential functions \( \psi^{(i)} (\rho, z) \) are expanded in terms of eigen functions \( \Psi^{(i)} (z) \) as

\[
\psi^{(i)} (\rho, z) = \sum_{n=0}^{\infty} \tilde{\psi}^{(i)} (\rho) \Psi^{(i)} (z)
\]

Eigen functions in regions (II) and (III) are the simple sinusoidal functions which satisfy the boundary conditions at the ring conductors at \( z = 0 \), and at the ground conductors \( z = h \) and \( z = -d \), respectively. On the other hand, the regions (I) and (IV) are inhomogeneous and fields in these regions cannot be expanded in terms of simple sinusoidal functions. The eigen functions \( \Psi^{(i)} (z) \), which fulfill the boundary conditions at the boundary \( z = 0 \) as well as at the ground conductors \( z = h \) and \( z = -d \), are derived by using the similar procedure used in [2]. The eigen functions \( \Psi^{(i)} (z) \) satisfy the biorthogonal relation [2], and fields in inhomogeneous regions (I) and (IV) as well as homogeneous regions (II) and (III) can be transformed into the spectral domain. The transformed potential functions can be expressed in general as

\[
\tilde{\psi}^{(i)} (\rho, n) = A I_0 (\alpha^{(i)}_n \rho) + B K_0 (\alpha^{(i)}_n \rho)
\]

where \( I_0 (\alpha^{(i)}_n \rho) \) and \( K_0 (\alpha^{(i)}_n \rho) \) are modified Bessel functions, and \( A \) and \( B \) are unknown coefficients, which can be expressed in terms of the aperture fields. Fields in each region, then, can be related to these aperture fields easily in the transformed domain. The unknown aperture fields are determined by enforcing the condition that the total electrostatic energy \( U \) stored in whole region to be minimum (Thomson theorem). The capacitance is then obtained as

\[
C = \frac{2U}{V_0^2} = \frac{2}{V_0^2} \sum_{n=0}^{\infty} \tilde{F} (\alpha^{(i)}_n) e^{(i)} (z) e^{(i)} (z)
\]

where \( \tilde{F} (\alpha^{(i)}_n) \) is the transformed field.
where $\tilde{F}(\alpha^{(i)}_k)$ and $\tilde{e}^{(i)}(z)$ are the Fourier transforms of Green’s function and the aperture fields, respectively. $V_0$ is the voltage between the ring and ground conductors, which can be obtained by integrating the electric static field over the aperture. It can be shown easily that eq.(5) has a stationary property and that it gives an upper bound of the exact value $C$.

2.2 Derivation of the lower bound expression of capacitance

The lower bound expression of capacitance can be obtained by using the charge density on the microstrip ring $\sigma(\rho)$ as the source quantity. The derivation is explained in [1],

$$\frac{1}{C} = \frac{1}{Q} \int_0^\infty \tilde{G}(\xi) \tilde{\sigma}(\xi) d\xi$$

where $\tilde{G}(\xi)$ and $\tilde{\sigma}(\xi)$ are the Hankel transforms of Green’s function and the charge density on microstrip ring, respectively, and $Q$ is the total charge on the ring. Eq.(6) has a stationary property and that it gives a lower bound of the exact value $C$.

3. Numerical Procedure and Results

The numerical procedure to obtain the capacitance is based on Galerkin’s method. The unknown source quantities, i.e., the aperture fields in the upper bound expression (5) and the charge density in the lower bound expression (6), are expanded in terms of the appropriate basis functions $f_k^{(i)}(z)$ and $f_k(\rho)$.

$$e^{(i)}_k(z) = \sum_{k=1}^{N} a_k^{(i)} f_k^{(i)}(z), \quad \sigma(\rho) = \sum_{k=1}^{N} b_k f_k(\rho)$$

where $a_k, b_k$ are the unknown expansion coefficients. For the accurate and efficient computation, the basis functions take the singularities of fields near the conductor edge into consideration. We choose the following basis functions

$$f_k^{(a)}(z) = \frac{T_{z(k-\frac{1}{2})} \left( \frac{z-h}{h} \right)}{\sqrt{1 - \left( \frac{z-h}{h} \right)^2}}$$

for the aperture fields [3], and

$$f_k(\rho) = \frac{T_{z(k-\frac{1}{2})} \left( \frac{\rho^2 - R^2}{(R+W)^2 - R^2} \right)}{\sqrt{\rho^2 - R^2} \left( (R+W)^2 - \rho^2 \right)}$$

(9)
for the charge density[4], where \(T_i(x)\) are Chebyshev polynomials. The similar basis functions are used for other aperture fields in the upper bound expression.

Fig. 2 shows the convergence of the capacitance with regard to the number of Basis functions \(N\). The value of upper bound expression is decrease as increases number of Basis function, and the value of lower bound expression is increase as increases number of Basis function, and the range of error becomes less than 0.002% for \(N = 5\). Fig. 3 shows the capacitance of radius \(R\) for different dielectric constants of substrates \(\varepsilon_r\). This figure also includes the values by finite element method (FEM). Excellent agreement is observed for all cases.

4. Conclusion

The accurate upper and lower bounds values evaluations are presented for the capacitance of annular ring by using the proper basis functions with edge singularity. The numerical computations define the range of error less than 0.002%.

ACKNOWLEDGEMENT

The work was supported in part by the Kinki Mobile Radio Center, Foundation under KMRC R&D Grant for Mobile Wireless.

References