Asymptotic Analysis Methods for Scattered Fields by a Coated Conducting Cylinder

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1. Introduction

The problems of the High-frequency (HF) scattering by coated conducting cylinder covered by a dielectric material have been an important research subject in the area of radar cross section, antennas and propagation, and so on [1], [2].

We have derived in [3] the extended uniform geometrical theory of diffraction (extended UTD) solution for the scattered fields by a coated conducting cylinder with a lossy medium. The extended UTD solution characterized by an impedance boundary condition (IBC) and effective in the transition region near the shadow boundary (SB) in [3] has agreed excellently with the exact solution when the thickness of coating medium is thin. However, the accuracy of the asymptotic solution in [3] deteriorates gradually as the thickness of coating medium becomes thick. In order to solve the above problem, it is necessary to newly take existence of the $\mathcal{G}^m$ th times reflected geometrical boundary ($\mathcal{GB}^m$) into consideration where the $\mathcal{GB}^m$ denotes the tangent line at the refraction point of a coating surface after reflected $m$ times on a conducting cylinder (see Fig.2).

In this paper, we study the asymptotic analysis methods taken into account the effect of the scattering phenomena inside a coating medium. Specifically, we derive both an extended UTD solution for a reflected-surface diffracted ray ($\mathcal{RSD}$) effective in the transition region near the $\mathcal{GB}^m$ and a reflected-geometrical ray ($\mathcal{RGO}$) solution in the lit region away from the $\mathcal{GB}^m$. The validity and the applicability of the asymptotic solutions derived here are confirmed by comparing with the exact solution obtained from the eigenfunction expansion [3], [4].

2. Formulation and Integral Representation for Scattered Fields

Figure 1 shows a surrounding medium 1 ($\varepsilon_1$, $\mu_1$) and a coated conducting cylinder of radius $a$ covered by a complex dielectric medium 2 ($\varepsilon_2$, $\mu_2$) of thickness $t (= a - b)$, and coordinate systems $(x, y, z)$ and $(\rho, \phi, z)$. We examine the two-dimensional problem assuming that the electric line source $Q(\rho, \phi, z)$ is placed parallel to the coated cylinder.

The integral representation for the scattered fields $E_2^z$ observed at a point $P(\rho, \phi) = (\rho, \phi)$, may be given by the following summation of the three kinds of integrals [4]:

$$E_2^z = E_{z1} + E_{z2} + E_{z3}$$

(1)

Here, $E_{z1}$ denotes the integral representing the direct ray before passing through a turning point (TP) and $E_{z2}$ is the integral including both the direct ray after passing through a TP and the scattering phenomena on the coating surface. While, $E_{z3}$ denotes the integral including the scattering phenomena inside the coating medium 2 and may be represented as follows:

$$E_{z3} = \sum_{p=1}^{\infty} E_{z3}^p$$

(2)

$$E_{z3}^p = \frac{i}{8} \int_{-\infty}^{\infty} \kappa_1 T_{12} T_{21} (\kappa_2^2 R_{22}^{p-1}) H_v^{(1)}(k_1 \rho') H_v^{(1)}(k_1 \rho) \exp(i \nu |\phi - \phi'|) d\nu$$

(3)

$$\kappa_1 = -\frac{H_v^{(2)}(k_1 a)}{H_v^{(1)}(k_1 a)} , \quad \kappa_2 = -\frac{H_v^{(1)}(k_2 a) H_v^{(2)}(k_2 b) + H_v^{(2)}(k_2 a) H_v^{(1)}(k_2 b)}{H_v^{(1)}(k_2 a) H_v^{(1)}(k_2 b)}$$

(4)
\[ T_{12} = 1 + R_{11}, \quad T_{21} = 1 + R_{22} \]  
\[ R_{11} = -\frac{\log H^{(2)}_\nu(k_2' a) - Z_3 \log H^{(2)}_{\nu'}(k_1 a)}{\log H^{(2)}_\nu(k_2' a) - Z_3 \log H^{(1)}_{\nu'}(k_1 a)}, \quad R_{22} = -\frac{\log H^{(1)}_\nu(k_2' a) - Z_3 \log H^{(1)}_{\nu'}(k_1 a)}{\log H^{(2)}_\nu(k_2' a) - Z_3 \log H^{(1)}_{\nu'}(k_1 a)} \]  

Here \( H^{(1)}_\nu(\cdot) \) and \( H^{(2)}_\nu(\cdot) \) denote the Hankel functions of the first and the second kinds [5], respectively, and the prime (') on the functions denotes the derivative with respect to the argument. \( k_1 = \omega (\epsilon_1 \mu_0)^{1/2} \) and \( Z_1 = (\mu_0 / \epsilon_1)^{1/2} \) are the wavenumber and the characteristic impedance of the medium 1 (the medium 2). Notation \( \epsilon_2^* \) denotes the complex dielectric constant of the material 2 and is defined by \( \epsilon_2^* = \epsilon_2 + i \sigma_2 / \omega \) where \( \sigma_2 \) is the conductivity. Notation \( p = 1, 2, \cdots \) in (2) and (3) denotes the number of reflection on the conducting cylinder \( \rho = b \). The time convention \( \exp(-i\omega t) \) is adopted and suppressed here.

### 3. Asymptotic Solutions Including Scattering Phenomena inside Coating Medium

Figure 2 shows the shadow boundary SB (= GB\(_{p=0} \)) and the once \( (p = 1) \) reflected geometrical boundary GB\(_{p=1} \) (GB\(_1 \)) where \( p \) denotes the number of reflection on the conducting cylinder \( \rho = b \). The surrounding medium 1 is divided into the lit and the shadow region by the GB\(_1 \). When the observation point is located in the lit region away from the GB\(_1 \), the once reflected-geometrical ray (RGO\(_{p=1} \) (RGO\(_1 \)) is observed. While, we observe the once reflected-surface diffracted ray (RSD\(_{p=1} \) (RSD\(_1 \)) when the observation point is located in the shadow region far away from the GB\(_1 \).

In this section, we will derive the asymptotic solutions including the scattering phenomena inside a coating medium 2.

#### 3.1 Extended UTD Solution for Reflected-Surface Diffracted Ray

In this section, from the integral \( E_{z,3}^p \) in (3), we will derive the extended UTD solution for the \( p \) th reflected-surface diffracted ray (RSD\(_p \)) applicable uniformly in the transition region near the GB\(_p \) and in the deep shadow region far away from the GB\(_p \).

In the shadow region, the main contribution to the integral \( E_{z,3}^p \) in (3) arises from the portion of the integration path near \( \nu = k_1 a \) in the complex \( \nu \)-plane. In this case, one may replace the functions \( H^{(1),(2)}_\nu(k_1 a) \) and \( H^{(1),(2)}_{\nu'}(k_1 a) \) by their Airy approximations [5] and the function \( H^{(1)}_\nu(k_1 x) \), \( x = \rho' \) or \( x = \rho \), by the Debye’s approximation [5], respectively, with the transformation from the complex \( \nu \)-plane to the complex \( r \)-plane via \( \nu = k_1 a + M \tau, \quad M = (k_1 a/2)^{1/3} \). Then by performing the straightforward manipulation, one may obtain the following extended UTD solution [4].

\[ E_{z,3}^p \sim G(k_1 L_1) \exp(ik_1 L_1 t + ik_1 \ell) (R_2)^\nu I(\xi) G(k_1 L_2) \]  
\[ G(k_1 L, 2) = i \frac{2}{4 \pi k_1 L_{1,2}} \exp(ik_1 L_{1,2} - i \pi/4) \]  
\[ I(\xi) = \frac{i 8 M^2}{Z_s \cos \theta_c} \int_{C_r} \exp \left\{ \left[ i \xi \tau + i \left( \frac{M^2}{2k_1 L_1} + \frac{M^2}{2k_1 L_2} + \frac{(2p) M^2}{2k_1^2 \cos \theta_c} \right) \right] \tau^2 \right\} \]  
\[ \left\{ - \left( w'_1(\tau) + q(\tau) w_1(\tau) \right) \right\}^{p+1}_0 \]  
\[ \left\{ - \left( w'_1(\tau) - q(\tau) w_1(\tau) \right) \right\}^{p+1}_0 \]  
\[ M = (k_1 a/2)^{1/3}, \quad R_2 = -1, \quad \xi = M(\theta - (2p) \psi), \quad Z_s = Z_2/Z_1, \]  
\[ \theta = \phi - \phi', \quad \psi = \cos^{-1}(k_1/k_2), \quad L_1 = \sqrt{\rho^2 - \alpha^2}, \quad L_2 = \sqrt{\rho^2 - \alpha^2}, \quad \ell = a(\theta - (2p) \psi), \quad L_t = (2p)(\cos \theta_c - \cos \theta_t) \]  

where notations \( w_i(\phi) \) and \( \theta_i \) denote respectively the incident angle and the refraction angle on the surface \( \rho = a \), and \( \theta_t \) is the incident angle to the conducting cylinder \( \rho = b \) (see Fig.2).

We have also shown in Fig.2 the propagation path of the once reflected-surface diffracted ray \( E_{z,3}^3 \) with \( p = 1 \) in (7)-(12). Notations \( L_1, L_t, \ell \) and \( L_2 \) may be interpreted as follows. \( L_1 (= Q \rightarrow \)
denotes the propagation distance (path) of the incident cylindrical wave which illuminates the surface diffraction point \( Q_1 \) from the source point \( Q \), \( L_1 (= Q_1 \rightarrow Q_2 \rightarrow Q_3) \) denotes the propagation distance (path), where \( Q_2 \) denotes the reflection point on the conducting cylinder \( p = b \), passing through the medium 2, \( \ell ( = Q_3 \rightarrow Q_4) \) the propagation distance (path) of the creeping wave along on the convex surface \( \rho = a \), and \( L_2 (Q_4 \rightarrow P_2) \) the propagation distance (path) from the diffraction point \( Q_4 \) to the observation point \( P_2 \). Notation \( G(k_1 L_{1,2}) \) in (8) is the 2-dimensional free space Green’s function and the integral \( I(\xi) \) in (9) may be interpreted as the term including the effect of scattering phenomena that occurs on the propagation path from the point \( Q_1 \) to the point \( Q_4 \).

### 3.2 Reflected-Geometrical Ray Solution

In this section, from the integral \( E_{z,3}^p \) in (3), we will derive in the \( p \) th reflected-geometrical ray (RGO\(_p\)) solution applicable in the deep lit region far away from the GB\(_p\).

In the deep lit region, the main contribution to the integral \( E_{z,3}^p \) in (3) arises from the portion of the integration path near \( \nu = k_1 a \) in the complex \( \nu \)-plane. One may replace all the Hankel functions in (3) by the Debye’s approximation [5] with the transformation from the complex \( \nu \)-plane to the complex \( \omega \)-plane via \( \nu = k_1 \sin \omega \). Then by applying the saddle point technique [6], one may obtain the \( p \) th reflected-geometrical ray (RGO\(_p\)) solution [4]. The reader may obtain the explicit RGO\(_p\) solution in [4].

### 4. Numerical Results and Discussions

In order to confirm the validity and the applicability of the asymptotic solutions derived in Section 3, we have calculated the scattered fields by a coated conducting cylinder illuminated by the incident electric-type cylindrical wave.

Figure 3 shows the scattered field strength vs. \( |\phi - \phi'| \) curves. The shadow boundary SB (= GB\(_0\)) is located at \( |\phi - \phi'| = 95.7^\circ \), and the region in which the \( p \) th reflected-surface diffracted ray RSD\(_p\) can be observed is shown by the notation \( \rightarrow \), in the figure. The asymptotic solution (\( \bullet \bullet \bullet \bullet \bullet \) open circles) is obtained by using the direct ray, the reflected ray on the surface \( \rho = a \), and up to 3 times reflected-geometrical ray RGO\(_{p=3}\) on the conducting cylinder \( p = b \). The asymptotic solution (\( \bullet \bullet \bullet \) closed circles) is obtained by using both the extended UTD solution for the surface diffracted ray (\( p = 0 \)) along the surface \( \rho = a \) and the extended UTD solution series for the RSD\(_p\) in (2). It is observed that the asymptotic solutions (\( \bullet \bullet \bullet \bullet \bullet \) and \( \bullet \bullet \bullet \) ) agree excellently with the exact solution (\( \rightarrow \) : solid curve) in each region.

Also shown in Fig.3 is the conventional GO solution [3] (the direct ray and the reflected ray on the surface \( \rho = a \) (---- : dashed curve) for the lit region and the conventional extended UTD solution [3] (----- : dashed curve) for the region \( 46.0^\circ \leq |\phi - \phi'| \leq 180.0^\circ \). The conventional GO solution agrees well in the lit region. However, the conventional extended UTD solution becomes inaccurate in the region \( 46.0^\circ \leq |\phi - \phi'| \leq 180.0^\circ \). It is clarified that the conventional extended UTD solution produces the large errors in the transition and the shadow region.

### 5. Conclusion

We have derived the extended UTD solution for the reflected-surface diffracted ray and the reflected-geometrical ray solution taken into account the effect of the scattering phenomena inside the coating medium of a coated conducting cylinder. The accuracy of the asymptotic solutions derived here has been confirmed by comparing with the exact solution.

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References

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Fig.1 Coated conducting cylinder, and coordinate systems $(x, y, z)$ and $(\rho, \phi)$. $Q$: electric line source, $P$: observation point.

Fig.2 Direct ray, once reflected-geometrical ray $RGO_1$, and once reflected-surface diffracted ray $RSD_1$. Also shown are the transition regions near the $SB (= GB_0)$ and $GB_1$.

Fig.3 Scattered fields by a coated conducting cylinder calculated from the asymptotic solutions and exact solution. The numerical parameters used in the calculation: $a = 5.0$, $k_1 a = 100$, $t = 2.0 \lambda$, $\varepsilon_1 = \varepsilon_0$, $\varepsilon_2^r = \varepsilon_0 \varepsilon_2 r$, $\varepsilon_2 t = 3 + i 0.1$, source point: $(\rho', \phi') = (7.0, 0.0^\circ)$ and observation point: $(\rho, \phi) = (8.0, \phi)$. $\cdots \cdots$: asymptotic solution including $RGO_p$, $\cdots \cdots$: asymptotic solution including $RSD_p$, $\cdots \cdots$: exact solution, $\cdots \cdots$: conventional asymptotic solutions.